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SIGNAL PROPAGATION IN A RELATIVISTIC FLUID WITH VISCOSITY AND HEAT CONDUCTION†

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In order to eliminate the paradox due to the faster-than-light propagation of signals in standard relativistic models of fluids with dissipation, it is proposed to replace the dissipative coefficients in the constitutive equations by relaxation kernels, i.e. to use a theory with memory. It is shown that this yields signals with finite velocity, which, however, need not be less than that of light. The condition that the signal propagate at a velocity not exceeding that of light in a vacuum imposes certain *a priori* restrictions on the dissipative characteristics of a fluid.

INTRODUCTION

FLUIDS or gases with viscosity and heat conduction are described in relativity theory by two standard models, due respectively to Eckart [1] and Landau and Lifshits [2]; these models are physically equivalent [3], both preserving the characteristic feature of the non-relativistic Navier–Stokes–Fourier model, namely infinitely fast signal propagation in a locally attached inertial frame of reference (LAIFR).

In non-relativistic mechanics the unrealiability of theories with instantaneous signal propagation has long been recognized. A modified heat equation was proposed, with the result that the dynamics of the temperature field is governed by a telegraph-type equation. This idea was generalized by postulating a relaxation connection between the heat flux and the temperature gradient [5], i.e. basing the discussion on a theory of media with memory [6, 7]. A non-relativistic model of a viscous heat-conducting fluid with memory was constructed and it was proved that the signal velocity in such

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a fluid is finite [8]. For infinitely slow processes, the constitutive equations of the model reduce to the Navier–Stokes and Fourier equations.

Instead of the theory of media with memory, some of those working in non-equilibrium thermodynamics have proposed a different approach, known as extended thermodynamics [9, 10], which also leads to finite signal velocities. This involves treating the dissipative fluxes as independent variables. From a formal, mathematical point of view, the equations that arise here are identical with those of the theory of media with memory for a particular form of the relaxation kernels. Nevertheless, advocates of extended thermodynamics consider their approach more fundamental (see the discussion in [10]).

Methods of relaxation hydrodynamics will be used below to construct a model of a viscous heat-conducting fluid within the framework of special relativity.

Throughout, the plane Minkowski metric $(g_{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1)$ will be used. Greek indices take values of 0, 1, 2 and 3, referring to some inertial frame of reference x^α, with x^0 playing the role of time. Latin indices a, b take values of 0, 1, 2 and 3, Latin indices i, j, k, l take values of 1, 2 and 3. Repeated indices indicate summation. In Secs 1 and 2 the formulas will be simplified by using a system of units of measurements in which the speed of light in vacuum is unity. Single-component fluids only will be considered and gravitation will be ignored.

1. Following Eckart [1], we define the 4-velocity of a medium so that the mass flux in the LAIFR is zero. If n is the particle number density of the medium in the LAIFR, then $j^\alpha = nu^\alpha$ is the mass flux 4-vector. The 4-velocity is normalized to give $u_\alpha u^\alpha = 1$ and the flux satisfies the conservation equation

$$j^\alpha{}_{,\alpha}=0 \tag{1.1}$$

Let ε be the energy density in the LAIFR, q^α the heat flux 4-vector and $\pi^{\alpha\beta}$ the symmetric stress tensor. The energy-momentum tensor of the medium is given by the formula

$$T^{\alpha\beta}=\varepsilon u^\alpha u^\beta+q^\alpha u^\beta+u^\alpha q^\beta+\pi^{\alpha\beta} \tag{1.2}$$

With this notation we have relations

$$q^\alpha u_\alpha=0, \quad \pi^{\alpha\beta}u_\alpha=0 \tag{1.3}$$

The motion of the medium obeys the equations of energy-momentum conservation:

$$T^{\alpha\beta}{}_{,\beta}=0 \tag{1.4}$$

We define the projection tensor as $\Delta^{\alpha\beta} = u^\alpha u^\beta - g^{\alpha\beta}$. Then

$$\Delta_{\alpha\gamma}\Delta^{\gamma\beta}=-\Delta_\alpha{}^\beta \tag{1.5}$$

We set

$$\pi^{\alpha\beta}=p\Delta^{\alpha\beta}-\tau^{\alpha\beta} \tag{1.6}$$

where p is the hydrostatic pressure and τ^{ab} is the tensor of viscous stresses. We set $D = u^\alpha \partial/\partial x^\alpha$, $h = \varepsilon + p$, the latter being the enthalpy density in the LAIFR. Then, substituting (1.2) into (1.4) and using (1.3), (1.5) and (1.6), we obtain the equations of motion

$$\begin{gathered}0=\Delta_{\alpha\gamma}T^{\gamma\beta}{}_{,\beta}=hDu_\alpha-\Delta_{\alpha\gamma}Dq^\gamma+q_\alpha u^\beta{}_{,\beta}+\\ +u_{\alpha,\beta}q^\beta+\Delta_\alpha{}^\gamma p_{,\gamma}+\Delta_{\alpha\gamma}\tau^{\gamma\beta}{}_{,\beta}\end{gathered} \tag{1.7}$$

Let T be the absolute temperature, s the entropy density in the LAIFR and μ the chemical potential. These quantities satisfy the thermodynamic relations

$$d\varepsilon=Tds+\mu dn, \quad \varepsilon=Ts-p+\mu n \tag{1.8}$$

It follows from (1.1) and (1.8) that

$$0=u_\alpha T^{\alpha\beta}{}_{,\beta}=T(u^\alpha s)_{,\alpha}-Du^\alpha q_\alpha+q^\beta{}_{,\beta}+\tau^{\alpha\beta}u_{\alpha,\beta}$$

This implies an equation for the production of entropy:

$$(su^\alpha+T^{-1}q^\alpha)_{,\alpha}-\sigma=0 \tag{1.9}$$

$$\sigma=q^\alpha[(T^{-1})_{,\alpha}+T^{-1}Du_\alpha]-T^{-1}\tau^{\alpha\beta}u_{\alpha,\beta} \tag{1.10}$$

where σ is the entropy produced per unit volume of the medium.

We will now consider a specific world line of a particle of the fluid: $x^\alpha = x^\alpha(\tau)$, where τ is the proper time. Let $e_a{}^\alpha = e_c{}^\alpha(\tau)$ be a tetrad carried along the world line in the sense of Fermi and Walker [11]. We have

$$e_0{}^\alpha(\tau)=u^\alpha(x^\beta(\tau)), \quad e_a{}^\alpha e_b{}^\beta g_{\alpha\beta}=g_{ab}.$$

We will need the following auxiliary notation:

$$n_a=n_{,\alpha}e_a{}^\alpha, \quad T_a=T_{,\alpha}e_a{}^\alpha, \quad p_a=p_{,\alpha}e_a{}^\alpha$$

$$u_{ab}=u_{\alpha,\beta}e_a{}^\alpha e_b{}^\beta, \quad q_a'=q_\alpha e_a{}^\alpha, \quad \tau_{ab}'=\tau_{\alpha\beta}e_a{}^\alpha e_b{}^\beta$$

Latin indices may be raised and lowered with the help of the metric g_{ab}. We note that by (1.3) $q_0' = 0$, $\tau_{0a}' = 0$.

Equations (1.1), (1.7) and (1.9) define the motion of the fluid. However, for the problem to be closed we also need the thermodynamic equations

$$p=p(n, T), \quad s=s(n, T) \tag{1.11}$$

and the constitutive equations, i.e., explicit expressions for the heat flux and the tensor of viscous stresses in terms of the density field n, the temperature T and the velocity u_α.

In the local theory, we set

$$q'^i(\tau_0)=q'^i(n(\tau_0), \quad T(\tau_0), \quad n_j(\tau_0), \quad T_j(\tau_0), \quad u_{jk}(\tau_0)) \tag{1.12}$$

$$\tau'^{ij}(\tau_0)=\tau'^{ij}(n(\tau_0), \quad T(\tau_0), \quad n_k(\tau_0), \quad T_k(\tau_0), \quad u_{kl}(\tau_0)) \tag{1.13}$$

$$\sigma(\tau_0)\geqslant 0 \tag{1.14}$$

at each point of the world line. Confining our attention to the linear approximation with respect to the spatial gradients in (1.12) and (1.13) and accordingly to the quadratic approximation for σ, we derive an approximate formula $Du_\alpha = -\Delta_\alpha{}^\gamma h^{-1}p_{,\gamma}$ from (1.7) and substitute it into (1.10). Then the constitutive equations (1.12) and (1.13) take the following special form [1]:

$$q_i'=-\varkappa(T_i-Th^{-1}p_i), \quad \tau_{ij}'=-\eta\lambda g_{ij}-2\mu s_{ij} \tag{1.15}$$

$$\lambda=u_i{}^i, \quad s_{ij}={}^1/_2(u_{ij}+u_{ji})-{}^1/_3\lambda g_{ij}$$

where $\varkappa$, η and μ are the non-negative thermal conductivity and coefficients of bulk viscosity and translational viscosity, respectively. If the actual functions

$$\varkappa=\varkappa(n, T), \quad \eta=\eta(n, T), \quad \mu=\mu(n, T)$$

are known, the motion of the relativistic fluid is now described by a closed system of equations.

Model (1.15) has a serious fault: it predicts infinitely fast signal propagation in the fluid. We therefore consider a theory with memory, following the ideas from [8].

When the fluid exhibits memory effects, the heat flux and the tensor of viscous stresses at any point of the world line become functionals of all previous states of a fluid particle:

$$\begin{gathered} q'^i(\tau_0)=q'^i[n(\tau\leqslant\tau_0), \ T(\tau\leqslant\tau_0), \ n_j(\tau\leqslant\tau_0), \ T_j(\tau\leqslant\tau_0), \ u_{jk}(\tau\leqslant\tau_0)) \\ \tau'^{ji}(\tau_0)=\tau'^{ij}[n(\tau\leqslant\tau_0), \ T(\tau\leqslant\tau_0), \ n_k(\tau\leqslant\tau_0), \ T_k(\tau\leqslant\tau_0), \ u_{kl}(\tau\leqslant\tau_0)) \end{gathered} \tag{1.16}$$

Let us suppose that as $\tau\to\pm\infty$ the fluid moves forward at constant velocity (as a rigid body). Then we have the Clausius–Duhem inequality, which generalizes condition (1.14):

$$W \geqslant 0, \quad W = \int_{-\infty}^{+\infty} \sigma \, d\tau \tag{1.17}$$

Let us confine our attention to the case in which the density n and temperature T remain within a small neighbourhood of certain constants n_* and T_*. If f is any function of the parameters n and T, we will write

$$f_* = f|_{n=n_*,\ T=T_*}$$

We will need auxiliary notation:

$$\delta = n - n_*, \quad \theta = T - T_*, \quad \xi = p - p_*, \quad \delta_a = \delta_{,\alpha} e_a{}^\alpha$$
$$\theta_a = \theta_{,\alpha} e_a{}^\alpha, \quad \xi_a = \xi_{,\alpha} e_a{}^\alpha, \quad \Theta_a = {}_a\theta - T_* h_*^{-1} \xi_a$$

We will confine our attention to the linear approximation of the functionals (1.6). The simplest linear functionals are convolutions with relaxation kernels. Proceeding as in [8], we take

$$\begin{gathered} q_i(\tau_0) = -\int_{-\infty}^{+\infty} K_3(\tau_0 - \tau)\,\Theta_i(\tau)\,d\tau \\ \tau_{ij}'(\tau_0) = -\int_{-\infty}^{+\infty} K_1(\tau_0 - \tau)\,\lambda(\tau)\,d\tau g_{ij} - 2\int_{-\infty}^{+\infty} K_2(\tau_0 - \tau)\,s_{ij}(\tau)\,d\tau \end{gathered} \tag{1.18}$$

It is natural to impose certain conditions on the relaxation kernels $K_A = K_A(\tau)$ $(A = 1, 2, 3)$. By the causality principle, $K_A(\tau) = 0$ if $\tau < 0$. When $\tau \geqslant 0$ the functions $K_A(\tau)$ are positive and monotone decreasing.

For any function of time $f(\tau)$ we let f_F denote the Fourier–Laplace transform of the function:

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} f(\tau)\,d\tau$$

If $f = f(\tau)$ is a real function, then

$$(f_F(\omega))^* = f_F(-\omega) \tag{1.19}$$

By the Paley–Wiener Theorem, $K_{AF}(\omega)$ are homomorphic functions in the half-plane $\operatorname{Im}\omega > 0$ and continuous up to the real axis.

For infinitely slow processes, the model (1.18) reduces to (1.15). The dissipative coefficients can be expressed in terms of the kernels:

$$\eta_* = \int_0^{+\infty} K_1(\tau)\,d\tau, \quad \mu_* = \int_0^{+\infty} K_2(\tau)\,d\tau, \quad \varkappa_* = \int_0^{+\infty} K_3(\tau)\,d\tau$$

We now return to condition (1.17). The following expression may be derived from (1.18):

$$\begin{gathered} W = \frac{1}{T_*}\int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 [K_1(\tau_1 - \tau_2)\,\lambda(\tau_1)\,\lambda(\tau_2) + 2K_2(\tau_1 - \tau_2)\,s^{ij}(\tau_1)\,s_{ij}(\tau_2) - \\ - T_*^{-1} K_3(\tau_1 - \tau_2)\,\Theta^i(\tau_1)\,\Theta_i(\tau_2)] \end{gathered}$$

Changing here to Fourier transforms we obtain

$$\begin{gathered} W = \frac{1}{\pi T_*}\int_0^{+\infty} d\omega\,[\operatorname{Re} K_{1F}(\omega)\,|\lambda_F(\omega)|^2 + 2\operatorname{Re} K_{2F}(\omega)\,s_F{}^{ij}(\omega)\,s_{ijF}^*(\omega) - \\ - T_*^{-1}\operatorname{Re} K_{3F}(\omega)\,\Theta_F{}^i(\omega)\,\Theta_{iF}^*(\omega)] \end{gathered}$$

Hence it is clear that the Clausius–Duhem inequality is equivalent to the inequalities $\operatorname{Re} K_{AF}(\omega) \geqslant 0$, $A = 1, 2, 3$.

If $\operatorname{Re} K_{AF}$ vanishes at some frequency, this implies the existence of a non-dissipative oscillatory process at that frequency—a phenomenon that may occur in superfluids. However, we shall disregard such cases and impose stronger conditions:

$$\operatorname{Re} K_{AF}(\omega) > 0, \quad A = 1,\ 2,\ 3 \tag{1.20}$$

We have the asymptotic formula

$$\begin{gathered} K_{AF}(\omega) = (i\omega)^{-1} K_A(0) + o(|\omega|^{-1}) \\ \operatorname{Im}\omega \leqslant 0, \quad A = 0,\ 1,\ 2,\ 3, \quad K_0(\tau) = K_1(\tau) + {}^4/_3 K_2(\tau) \end{gathered} \tag{1.21}$$

It follows from this formula and the properties of holomorphic functions that inequalities (1.20) hold throughout the lower complex half-plane.

2. Let us consider the propagation of small disturbances in a fluid at rest. We will use the notation:

$$\nabla = \mathbf{grad}, \quad \nabla^+ = \mathbf{div}, \quad \Delta = \sum_{i=1}^{3} \left(\frac{\partial}{\partial x^i} \right)^2$$

$$p_{n*} = \left(\frac{\partial p}{\partial n} \right)_{T_*}, \ p_{T*} = \left(\frac{\partial p}{\partial T} \right)_{n_*}, \ s_{n*} = \left(\frac{\partial s}{\partial n} \right)_{T_*}, \ s_{T*} = \left(\frac{\partial s}{\partial T} \right)_{n_*}$$

$\mathbf{v}$ is a column 3-vector with $v_i = v^i$ and $g_1 * g_2$ is the convolution of two functions with respect to time.

Since we are concerned here with small perturbations, we may assume that the integration in (1.18) is performed with respect to the time x^0 at a point of space with fixed coordinates x^i.

Then Eqs (1.1), (1.7) and (1.9) become

$$\begin{gathered} \frac{\partial \delta}{\partial x^0} + n_* \nabla^+ \mathbf{v} = 0 \\ h_* \frac{\partial \mathbf{v}}{\partial x^0} + p_{n*} \nabla \delta + p_{T*} \nabla \theta - \left(K_1 + \frac{1}{3} K_2 \right) * \nabla \nabla^+ \mathbf{v} - K_2 * \Delta \mathbf{v} + \\ + \frac{\partial}{\partial x^0} \nabla \left[K_3 * (T_* h_*^{-1} p_{n*} \delta + (T_* h_*^{-1} p_{T*} - 1) \theta) \right] = 0 \\ \frac{\partial}{\partial x^0} (s_{n*} \delta + s_{T*} \theta) + s_* \nabla^+ \mathbf{v} + T_*^{-1} \Delta \left[K_3 * (T_* h_*^{-1} p_{n*} \delta + (T_* h_*^{-1} p_{T*} - 1) \theta) \right] = 0 \end{gathered} \tag{2.1}$$

We will denote the Fourier transform of an arbitrary function $g = g(x^\alpha)$ by the symbol $g_\Phi(\omega, k_i)$:

$$g_\Phi(\omega, k_i) = \int e^{-i(\omega x^0 + k_j x^j)} g(x^\alpha)\, dx^0\, dx^1\, dx^2\, dx^3$$

We will now let the superscript $+$ denote matrix transposition, and the symbol k the column vector k_i, $k^2 = k^+ k$. Carrying out a Fourier transformation we convert Eqs (2.1) into a system of linear algebraic equations:

$$\Sigma \begin{Vmatrix} \delta_\Phi \\ \theta_\Phi \\ v_\Phi \end{Vmatrix} = 0, \quad \Sigma = \Sigma(\omega, k) =$$

$$= \begin{Vmatrix} i\omega & 0 & i n_* \mathbf{k}^+ \\ i(1 + i\omega K_{3F} T_* h_*^{-1}) p_{n*} \mathbf{k} & \begin{matrix} i(p_{T*} + i\omega K_{3F} \times \\ \times (T_* h_*^{-1} p_{T*} - 1)) \mathbf{k} \end{matrix} & \begin{matrix} i\omega h_* + (K_{1F} + {}^1/_3 K_{2F}) \mathbf{k}\mathbf{k}^+ + \\ + K_{2F} k^2 \end{matrix} \\ i\omega s_{n*} - K_{3F} h_*^{-1} p_{n*} k^2 & \begin{matrix} i s_{T*} \omega + K_{3F} \times \\ \times (T_*^{-1} - h_*^{-1} p_{T*}) k^2 \end{matrix} & i s_* \mathbf{k}^+ \end{Vmatrix} \tag{2.2}$$

To investigate the propagation of a signal from some source, it is generally necessary to add sources of matter, force and entropy on the right of Eqs (1.1), (1.7) and (1.9) [and, accordingly, (2.1)]. We will assume that the sources are concentrated at a point x^α in space-time. We must then add a constant 5-component vector $\mathbf{l}$ to the right-hand side of Eq. (2.2), to describe the strength of the sources.

Consider the perturbations of the density, temperature and velocity fields induced by the action of the sources at a point of space with coordinates $x_0{}^i = L\delta_1{}^i$, $L>0$.

After some calculation we get

$$\begin{Vmatrix} \delta_\Phi(\omega, x_0) \\ \theta_\Phi(\omega, x_0{}^j) \\ \mathbf{v}_\Phi(\omega, x_0{}^j) \end{Vmatrix} = \int_{-\infty}^{+\infty} e^{-i\omega x^0} \begin{Vmatrix} \delta(x^0, x_0{}^j) \\ \theta(x^0, x_0{}^j) \\ \mathbf{v}(x^0, x_0{}^j) \end{Vmatrix} dx^0 = Y\mathbf{l} \tag{2.3}$$

$$Y = \frac{1}{(2\pi)^3}\int e^{ik_1L}\Big(\sum(\omega, \mathbf{k})\Big)^{-1} dk_1\, dk_2\, dk_3 \tag{2.4}$$

We note that $(\Sigma(\omega, \mathbf{k}))^{-1} = P^{-1}\Sigma^*(\omega, \mathbf{k})$, where Σ^* is the complementary matrix to Σ, and

$$P = \det\Sigma(\omega, \mathbf{k}) = P_1{}^2P_2$$

$$P_1 = i\omega h_* + K_{2F}k^2$$

$$\begin{aligned} P_2 = P_2(\omega, k^2) = {} & (k^2)^2K_{3F}[-n_*p_{n*}T_*{}^{-1} + i\omega K_{0F}(h_*{}^{-1}p_{T*} - T_*{}^{-1})] + \\ & + k^2 i\omega[-s_{T*}n_*p_{n*} + p_{T*}(n_*s_{n*} - s_*) - s_{T*}i\omega K_{0F} + \\ & + i\omega K_{3F}(n_*p_{n*}s_{T*}T_*h_*{}^{-1} - h_*T_*{}^{-1} + p_{T*} + \\ & + (T_*h_*{}^{-1}p_{T*} - 1)(s_* - n_*s_{n*}))] + is_{T*}h_*\omega^3 \end{aligned}$$

It is convenient to carry out the integration in (2.4) in spherical coordinates R, ψ, φ, related to k_1, k_2, k_3 by the equations

$$k_1 = R\sin\psi\cos\varphi, \qquad k_2 = R\sin\psi\sin\varphi, \qquad k_3 = R\cos\psi$$

$$R \geqslant 0, \qquad \psi \in [-\pi/2, \pi/2], \qquad \varphi \in [-\pi, \pi]$$

After integration with respect to φ, every element of the matrix Y is the sum of terms of the form

$$y = C(\omega)\int\limits_{\substack{0 \leqslant R < +\infty \\ -\pi/2 \leqslant \psi \leqslant \pi/2}} \exp(iRL\sin\psi)\, R^{n+2}\sin^l\psi\cos^{m+1}\psi\,(P(\omega, R^2))^{-1}\, dR\, d\psi \tag{2.5}$$

Since the elements of the matrix $\Sigma^*(\omega, k)$ are polynomials in k_i, it follows that $n = l+m$. The matrix $\Sigma(\omega, k)$ [and hence also $\Sigma^*(\omega, k)$] remains invariant under the formal substitution $(R, \psi, \varphi) \mapsto (-R, -\psi, -\varphi)$, whence it follows that $m = 2q$, where q is a natural number.

We now substitute $w = \sin\psi$ into (2.5) and integrate with respect to w:

$$\begin{aligned} y &= C(\omega)\sum_{d=0}^{q} C_q{}^d(-1)^d \int\limits_{\substack{0 \leqslant R < +\infty \\ -1 \leqslant w \leqslant 1}} \exp(iRLw)\, R^{l+2q+2}w^{l+2q}(P(\omega, R^2))^{-1}\, dR\, dw = \\ &= C(\omega)\sum_{d=0}^{q} a_d \int_{-\infty}^{+\infty} R^{2(q-d)+1}\exp(iRL)(P(\omega, R^2))^{-1}\, dR \end{aligned} \tag{2.6}$$

where $C_q{}^d$ are the binomial coefficients and a_d are constants. The last integral can be evaluated via the Residue Theorem, provided we close the path of integration in the upper complex half-plane. To that end we investigate the roots of the equation

$$P(\omega, R^2) = 0 \tag{2.7}$$

in R. If $\operatorname{Im}\omega \leqslant 0$ this equation clearly has six roots, and the set of roots is invariant under inversion $R \mapsto (-R)$. It is convenient to number the roots so that $\pm R_1(\omega)$ are the roots of the equation $P_1(\omega, R^2) = 0$, where $\operatorname{Im}R_1(\omega) \geqslant 0$, while $\pm R_2(\omega)$ and $\pm R_3(\omega)$ are the roots of $P_2(\omega, R^2) = 0$ where $\operatorname{Im}R_A(\omega) \geqslant 0$ $(A = 2, 3)$.

As ω varies in the lower complex half-plane, the roots $R_A(\omega)$ move in the upper complex half-plane. It is important that the roots never cross the real axis, i.e. $R_A = R_A(\omega)$ are smooth (even holomorphic) functions.

To prove this assertion, let R be an arbitrary non-zero real number. Set $F_1(\omega) = P_1(\omega, R^2)$. Then $F_1(\omega)$ is holomorphic in the lower complex half-plane. Consider the behaviour of this function on the boundary of its domain of definition. As $|\omega| \to \infty$, $\operatorname{Im}\omega < 0$, we have $\operatorname{Re} F_1 \to -h_* \operatorname{Im}\omega > 0$. As $\operatorname{Im}\omega \to 0$, we see from (1.20) that $\operatorname{Re} F_1 \to R^2 \operatorname{Re} K_{2F} > 0$. Hence it follows that $\operatorname{Re} F_1 > 0$ throughout the lower complex half-plane, and therefore $\operatorname{Im} R_1(\omega) > 0$ for $\operatorname{Im}\omega \leqslant 0$.

We now introduce a new thermodynamic variable $r = s/n$—the entropy per particle of the medium. Then the equation may be replaced by the equations $p = p(n, r)$, $T = T(n, r)$.

Put

$$p_{n*}' = \left(\frac{\partial p}{\partial n}\right)_{r*}, \quad p_{r*}' = \left(\frac{\partial p}{\partial r}\right)_{n*}, \quad T_{n*}' = \left(\frac{\partial T}{\partial n}\right)_{r*}, \quad T_{r*}' = \left(\frac{\partial T}{\partial r}\right)_{n*}$$

We obtain

$$P_2(\omega, R^2) = -is_T h_* \omega R^4 (z^2 + (\lambda_1 + \lambda_2 + \lambda_3) z + \lambda_1 \lambda_2)$$

$$z = i\omega/R^2, \quad \lambda_1 = n_*^{-1} \alpha K_{3F}, \quad \alpha = T_*^{-1} T_{r*}' - h_*^{-1} p_{r*}'$$

$$\lambda_2 = h_*^{-1} [K_{0F} + n_* T_*^{-1} T_{r*}' p_{n*}' \alpha^{-1} (i\omega)^{-1}]$$

$$\lambda_3 = n_* h_*^{-1} T_* \beta K_{3F} + n_* h_*^{-2} p_{n*}' p_{r*}' \alpha^{-1} (i\omega)^{-1} = \\ = b\lambda_1 + a(i\omega)^{-1}$$

$$\beta = T_*^{-1} T_{n*}' - h_*^{-1} p_{n*}', \quad b = n_*^2 h_* \alpha \beta^{-1} T_*^{-1}$$

$$a = n_* h_*^{-2} p_{n*}' p_{r*}' \alpha^{-1}$$

Let us assume the validity of the thermodynamic inequalities

$$p_{n*}' > 0, \quad p_{r*}' > 0, \quad \alpha > 0, \quad \beta > 0$$

Then, using conditions (1.20), we can show that if $\operatorname{Im}\omega < 0$

$$\operatorname{Re} \lambda_A > 0, \quad A = 1, 2, 3 \tag{2.8}$$

In addition, the following asymptotic equalities are obtained from (1.21):

$$\lambda_A(\omega) = \Lambda_A (i\omega)^{-1} + o(|\omega|^{-1}), \quad A = 1, 2, 3 \tag{2.9}$$

$$\Lambda_1 = n_*^{-1} \alpha K_3(0), \quad \Lambda_2 = h_*^{-1} (K_0(0) + n_* T_*^{-1} T_{r*}' p_{n*}' \alpha^{-1}), \quad \Lambda_3 = b\Lambda_1 + a$$

For real $R \neq 0$ we define

$$F_2(\omega) = z + \lambda_2 + z\lambda_3/(z + \lambda_1)$$

This function is holomorphic in the lower complex half-plane. We investigate its behaviour on the boundaries of its domain of definition. As $|\omega \to| + \infty$, $\operatorname{Im}\omega < 0$,

$$\operatorname{Re} F_2 \to -R^{-2} \operatorname{Im}\omega > 0$$

As $\operatorname{Im}\omega \to 0$,

$$\operatorname{Re} F_2 \to \operatorname{Re} \lambda_2 + R^{-2} \operatorname{Re} \frac{\lambda_3 i\omega}{z + \lambda_1}$$

If $\operatorname{Im}\omega = 0$ the last term becomes

$$R^{-2}(a + \omega^2 b R^{-2}) \operatorname{Re} \lambda_1 [(\operatorname{Re} \lambda_1)^2 + (\omega R^{-2} + \operatorname{Im} \lambda_1)^2]^{-1}$$

Hence, by (2.8), it follows that $\operatorname{Re} F_2 > 0$ as $\operatorname{Im}\omega \to 0$, and therefore $\operatorname{Re} F_2 > 0$ throughout the lower complex half-plane. It is now obvious that $\operatorname{Im} R_A(\omega) > 0$, $A = 2, 3$. $\operatorname{Im}\omega \leqslant 0$.

Integrating in (2.6), we obtain

$$y = \sum_{A=1}^{3} c_A(\omega) \exp(iR_A(\omega) L) \tag{2.10}$$

where $c_A(\omega)$ are holomorphic for $\operatorname{Im}\omega < 0$ and satisfy the inequalities

$$|c_A| \leqslant C|\mathrm{Im}\,\omega|^{-N_1}(1+|\omega|)^{N_2} \tag{2.11}$$

with certain positive constants C, N_1 and N_2.

Consider the functions

$$L_A = L_A(\omega) = \mathrm{Im}\,R_A(\omega)/(-\mathrm{Im}\,\omega), \qquad \mathrm{Im}\,\omega<0 \qquad (A=1,\ 2,\ 3)$$

It follows from (2.3) and (2.10) that

$$\left\|\begin{matrix}\delta_F(\omega, x_0^j)\\ \theta_F(\omega, x_0^j)\\ \mathbf{v}_F(\omega, x_0^j)\end{matrix}\right\| = \sum_{A=1}^{3} l_A \exp(iLR_A(\omega)) \tag{2.12}$$

where $l_A = l_A(\omega)$ are 5-component vectors, each component of which satisfies inequalities of type (2.11). It follows from (2.12) and the Paley–Wiener Theorem that there will be no disturbances at the point x_0^j for $x^0 < L$ if and only if

$$L_A(\omega) \geqslant 1, \qquad \mathrm{Im}\,\omega<0 \qquad (A=1,\ 2,\ 3)$$

Put

$$V_A^{-1} = \inf_{\mathrm{Im}\,\omega<0} L_A(\omega) \qquad (A=1,2,3)$$

We note that the functions $L_A = L_A(\omega)$ cannot attain their infima at any point of the lower complex half-plane.

Indeed, suppose the contrary: let $\omega = \omega_0$, $\mathrm{Im}\,\omega_0 < 0$ be a point at which $L_A(\omega)$ has an absolute minimum for some A, $L_A(\omega_0) = V_A^{-1}$. Then the following harmonic function of the two real variables ω_1 and ω_2

$$h_A = h_A(\omega_1, \omega_2) = \mathrm{Im}\,R_A(\omega_1 + i\omega_2) + V_A^{-1}\omega_2$$

assumes an absolute minimum zero at ω_0. Therefore $h_A \equiv 0$, contrary to our previous results.

Thus, we can write

$$V_A^{-1} = \lim_{\zeta\to+\infty} \inf_{|\omega|\geqslant\zeta} L_A(\omega)$$

Using (2.9), we now find that

$$V_1 = (h_*^{-1}K_2(0))^{1/2} \tag{2.13}$$

$$V_{2,3} = 2^{-1/2}(\Lambda_0 \pm (\Lambda_0^2 - 4\Lambda_1\Lambda_2)^{1/2})^{1/2}, \quad \Lambda_0 = \sum_{A=1}^{3}\Lambda_A$$

It is clear from the preceding arguments that V_A $(A = 1, 2, 3)$ are the maximum velocities of propagation of different modes in the fluid. Formula (2.13) agrees with our previous result [12], since this mode describes vortex transport in a relativistic fluid.

We have thus shown that the model of a relativistic fluid with memory provides a natural description of viscosity and heat conduction, while at the same time predicting a finite signal propagation velocity. For relaxation kernels of the most general form that satisfy all the conditions listed in Sec. 1, the velocity may nevertheless exceed the speed of light in a vacuum. The conditions $V_A \leqslant 1$ should be considered as certain *a priori* restrictions on the dissipative properties of the fluid.

The model proposed above should be used when the internal relaxation time of the medium is greater than or comparable in magnitude with the characteristic durations of the macroscopic processes. Otherwise the model will produce the same results as the Eckart or Landau–Lifshits models.

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FAST ASYMPTOTIC FORM OF THE RESISTANCE OF BODIES IN A WAVEGUIDE LAYER OF NON-UNIFORM FLUIDS†

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The asymptotic dependence of the wave resistance of bodies moving horizontally at a high speed in a waveguide with an arbitrary stratified fluid is analysed. For a waveguide of finite depth, it is established that the resistance is inversely proportional to the square of the velocity and directly proportional to the square of the volume, for small bodies. The general results are refined for uniform stratification and a pronounced transition layer.

WHEN bodies move in a density-stratified fluid internal waves are excited and perturbations propagate inside the fluid. By virtue of this fact, even if viscous resistance is neglected (in the ideal-fluid approximation) the body will experience wave resistance. It is convenient, when calculating this, to replace the boundary-value problem of the flow around the body by the problem of the motion of mass or force sources, which are equivalent to the body in their hydrodynamic effect on the fluid. These might be mass dipole sources, distributed over the surface of the submerged body and found from the solution of the boundary integral equations, for example. The use of model distributions of sources is especially helpful because it enables a number of general conclusions to be drawn without having to solve the quite time-consuming problem of the specific form of the source distributions.

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